On the Frictional Resistances to the Flow of Air through a Pipe.

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1. *Introduction.*

The research described in this paper was commenced by Dr. Grindley at the suggestion of Professor Osborne Reynolds in the Whitworth Engineering Laboratories, the Owens College, Manchester. Mr. Gibson continued the research after Dr. Grindley had left Manchester.

The experiments were commenced with the object of determining, on a larger experimental scale than usual, the coefficient of viscosity of air and other gases and the variation of this coefficient with temperature.

The present research, however, developed into a more complete investigation of the resistances to the passage of air through a pipe. The velocities of flow of the air through the pipe in the experiments by Dr. Grindley were for the most part above the critical velocity, the critical velocity being the velocity of flow below which the motion is steady or stream line and above which the motion becomes eddying or turbulent, the laws of resistance to the passage of the air being different in the two kinds of flow.

These experiments by Dr. Grindley indicated the value of the critical velocity of flow through the particular pipe used in the experiments, but the few experiments at velocities of flow below the critical were insufficient to permit any deductions on viscosity to be made, and it was left to Mr. Gibson to make all the experiments at velocities of flow below the critical, from the results of which the coefficient of viscosity has been deduced in this paper.

The scope of the research may best be shown by the following summary of some of the more important deductions.

(1) The coefficient of viscosity, μ, of dry air at 0° C. is 0·0001702 in C.G.S. absolute units.

(2) The law of variation of μ with temperature has been determined between the temperature limits 0° C. and 100° C., and is found to be represented by an equation of the form

$$\mu = \frac{\mathrm{K}\tau^{\frac{1}{2}}}{1+\mathrm{CT}^{-1}},$$

where K and C are constants, and τ is the absolute temperature.

(3) The absence of dependence of μ on the density of the air has again received experimental verification.

(4) The critical velocity for a small lead pipe—about 1/8 of an inch diameter—has been determined.

(5) In the turbulent motion of air through a pipe, the resistance has been found to be independent of the pressure of air in the pipe, and to be proportional to (velocity)^n, where $n = 1·25$ for this pipe.

(6) The law of variation of this resistance with temperature has been determined.

2. *Description of the Appliances.*

The general arrangement of the apparatus is shown in fig. 1. It consists of two gasholders A and B each of a little over 3 cubic feet capacity and connected by a length of lead tubing of roughly 1/8 of an inch internal diameter. Part of this length is to be used as the experimental tube, and at the ends of this, arrangements for measuring the pressures are provided.

The gasholder A contains air which can be forced through the experimental tube and into B by the simple expedients of admitting water into the lower part of A and emptying B of water contained by it at the beginning of each experiment.

These vessels A and B were calibrated so that the volume of air leaving or entering in any time interval could be obtained at once from the graduated gauges *aa* and *bb* by observing the heights of the columns of water in these tubes.

The length of tube between A and B is practically all wound on a central cylinder of brass D, the tube being wound at a constant small tension and fitting into a properly turned helical groove on the outer surface of the drum, the groove being just wide enough to take the tube, and about 5/32 of an inch deep, the pitch of the helix being one-third of an inch.

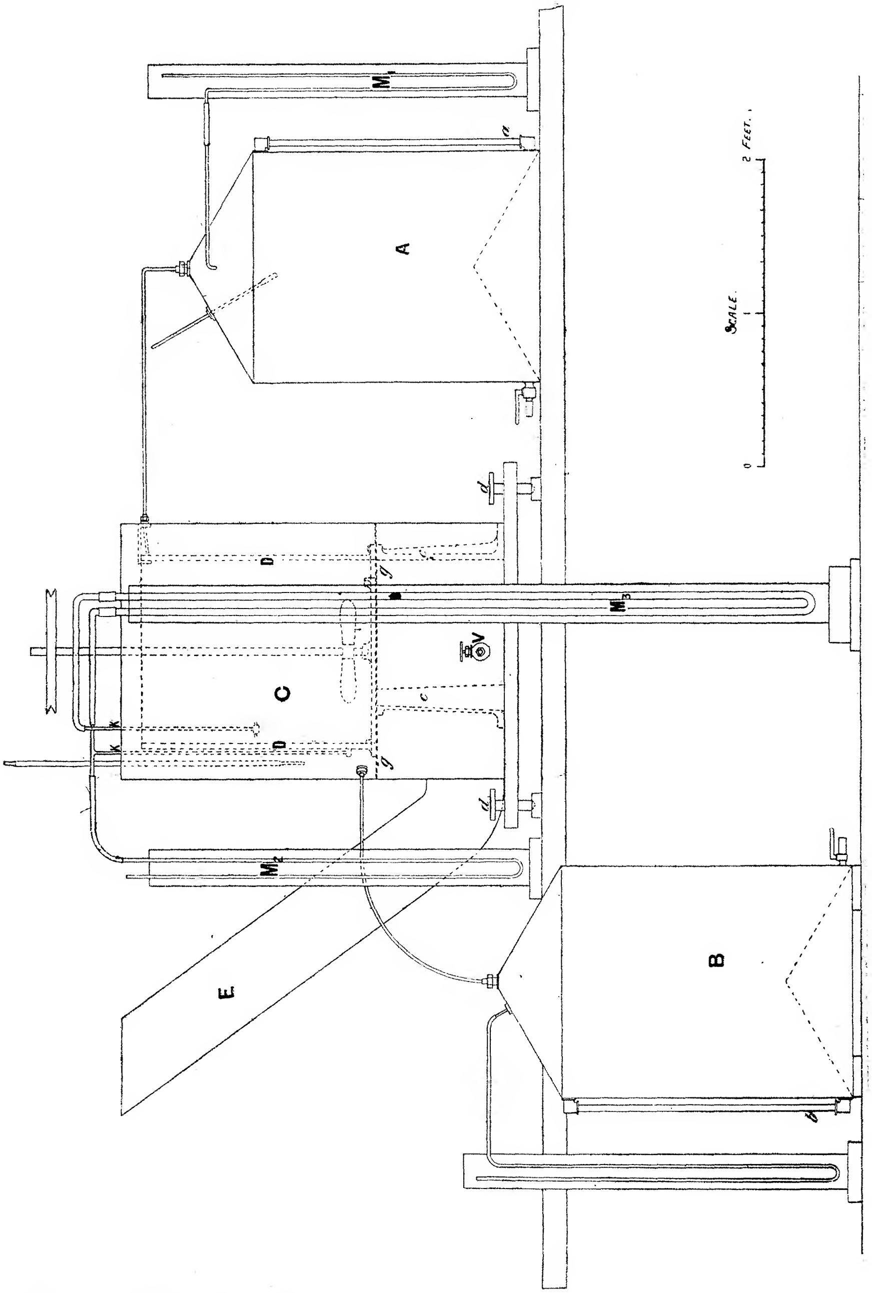

Fig. 1.

This cylinder D rested on three supports *c*, *c*, *c*, so as to occupy the upper central portion of a large circular vessel C of diameter 20 inches and height

30 inches. The axis of the cylinder D was made vertical by means of the three levelling screws *dd* in the table supporting the vessel C. For making experiments at different temperatures, the vessel C was filled with water, which could be raised or lowered to any desired temperature by admitting steam through the valve V or by introducing ice into the lower part of C by the side chute E. A wire gauge *gg* for keeping the ice in the lower part of the vessel, a screw stirrer and a thermometer were also provided in vessel C.

The total length of lead tube inside the vessel C was about 190 feet, of which about 75 feet, or 20·1 coils, were used to change the temperature of the air which is flowing through the tube to that of the surrounding water before it passes into the experimental part, which consists of about 108 feet of tubing, or over 29 coils. From the end of the experimental tube a further 8 feet of tube passed the air into the gasholder B.

The pressures in A and B and at the ends of the experimental tube are required, and those for vessels A and B were obtained by two mercury manometer gauges, M_1 and M_2. The pressures at the ends of the experimental tube required to be taken very accurately, and they were obtained in the following manner:—At each end of the experimental tube, the tube was raised out of its groove on cylinder D, as shown in fig. 2, G and G being two brass grooved pieces fastened to the drum D. H is also a brass piece fastened to the drum D, but having a central hole of diameter equal to the mean internal diameter of the lead tube. From this passage the pressure of

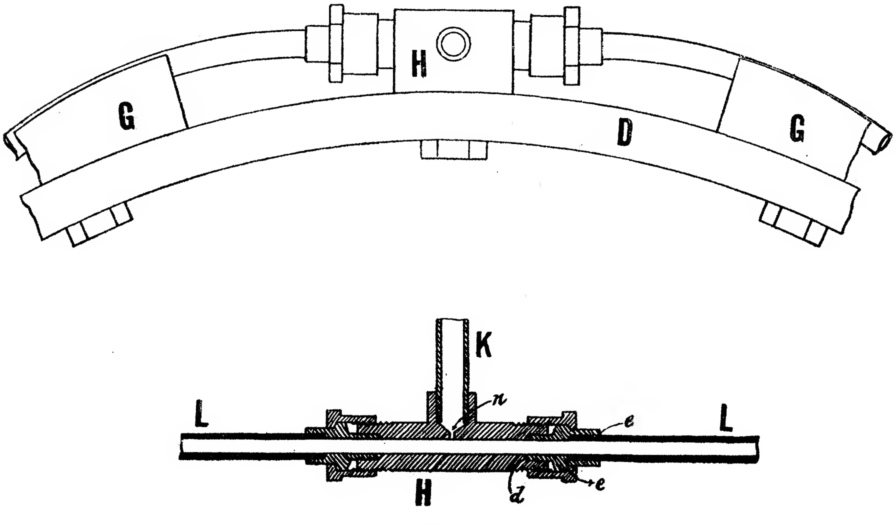

FIG. 2.

the air is taken by means of the small hole *n*, leading to the vertical tube K (shown in fig. 1 and partly in fig. 2), which in turn communicates with the pressure gauges. As shown in fig. 2, the experimental tube LL enters the flanged nipple R, and is soldered to this at Q.

The pressures are read from the vertical pressure tubes KK as follows:—The low pressure end of the experimental tube is in communication with two gauges, one (a mercury gauge) giving the actual pressure at this point, while the second (a water gauge with inclined scale), M_3, was also in communication with the high pressure end of the tube, and thus recorded the pressure difference along the experimental tube.

It is estimated that the limit of accuracy attainable with this pressure gauge was in the neighbourhood of 1 in 250, and in consequence the results of any experiment cannot claim any further degree of accuracy than about one-half of 1 per cent.

In all experiments carried out below the critical velocity, drying tubes were placed below the gasholder A and the experimental tube.

For much assistance in making the apparatus, the authors are indebted to Mr. T. Foster, of the Whitworth Engineering Laboratories at the Manchester University.

3. *Theoretical Considerations.*

When the velocity of flow of the air through the pipe is less than the critical velocity, the equation connecting the flux and the fall of pressure is easily deduced* from the definition of the coefficient of viscosity. It appears that if a is the inner radius of the pipe, l its length, and p_0-p_1 the difference of the pressures at the ends of the experimental pipe, the coefficient of viscosity is given in absolute measure by

$$\mu = \frac{\pi a^4}{8l} \cdot \frac{p_0-p_1}{w}, \tag{1}$$

where w is the volume of air flowing across any section of the pipe per second. In the experiments, w is taken as the quantity of air passing through the tube per second at the mean pressure $\frac{1}{2}(p_0+p_1)$ of the air in the pipe.

When the velocities of flow are greater than the critical, the form of the equations of motion depends on the assumptions made in the estimation of the resistances. The usually accepted law for the resistance R_1 to the passage of a fluid moving with mean velocity u over a surface of area Δ is given in gravitational units by the equation

$$R_1 = \zeta \frac{D}{2g} \cdot u^2 \Delta, \tag{2}$$

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* See Lamb, 'Hydrodynamics,' 3rd edit., p. 544.

where D is the weight per unit of volume of the fluid, and ζ is a coefficient usually termed the coefficient of friction.

The chief assumptions here made are that the resistance varies with the density of the fluid and with the square of its mean velocity. With compressible fluids, however, the laws of resistance represented by equation (2) have been assumed without adequate experimental basis. On integrating the equations of motion for the air in the pipe, assuming these laws of resistance, we are led to the following equation for the initial velocity u_0 of the air entering the experimental pipe :

$$u_0^2 = g\mathrm{R}\tau\,(p_0^2-p_1^2)/p_0^2\left(\frac{\zeta l}{m}+\log_e\frac{p_0}{p_1}\right),\text{*} \tag{3}$$

where R is the constant in the characteristic equation for air $pv = \mathrm{R}\tau$, p_0 and p_1 the pressures at the beginning and end of the experimental pipe, l the length of the tube, and m its hydraulic mean depth. If the difference of pressure p_0-p_1 is relatively small, the term $\log_e p_0/p_1$ is usually neglected, and equation (3) written

$$u_0^2 = gm\mathrm{R}\tau\,(p_0^2-p_1^2)/\zeta l p_0^2. \tag{4}$$

This equation (4) is applied in this paper to the experimental results, but the values of ζ, calculated from the experimental numbers, showed such wide variations that it was thought advisable to test the assumptions embodied in (2) if possible.

If a general equation for the resistances R_1 to the motion of any gas over any surface of area Δ were constructed, it would include factors for possible variations of R_1 with the pressure, temperature, and the velocity of the gas, and with the shape, dimensions, and state of the tube's inner surface. Since the variation in density in any experiment at constant temperature has a corresponding pressure variation, we may assume for each experiment an equation for the resistance to motion in the form

$$\mathrm{R}_1 = f\Delta p^r u^n, \tag{5}$$

where the coefficient f should be independent of the pressure p and velocity u of the gas, though it may vary (*a*) with the nature of the gas, (*b*) with the temperature, and (*c*) with the state of the tube's inner surface.

Equation (2) is a special case of (5) with

$$n = 2, \quad r = 1, \quad \text{and} \quad f = \frac{\zeta}{2g\mathrm{R}\tau}, \tag{6}$$

since $\mathrm{D} = p/\mathrm{R}\tau$ in equation (2).

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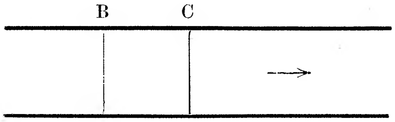
* See 'Encycl. Brit.,' art. "Hydraulics," vol. 15, p. 491.

To form the equations of motion, using (5), let

S = perimeter of cross-section of tube;
A = its uniform area;
W = weight of gas passing any section of the tube per second;
p = mean pressure across any cross-section B;



$p+\delta p$ = mean pressure across any section C distant δx from B; and consider the forces on the mass of gas between B and C; the principle of momentum would give at once, u being the mean steady velocity across the section of the tube,

$$-\mathrm{A}\delta p = \frac{\mathrm{W}}{g}\delta u + f\mathrm{S}p^r u^n \delta x. \tag{7}$$

But $\mathrm{W} = \mathrm{A}u/v$, when v is the specific volume of the gas, hence $\mathrm{W} = \mathrm{A}up/\mathrm{R}\tau$, and since the temperature is constant during any experiment, we have $pu = p_0u_0$, so that

$$\mathrm{W} = p_0u_0\mathrm{A}/\mathrm{R}\tau. \tag{8}$$

By substitution in (7) and dividing throughout by $\mathrm{A}p^ru^n$ we obtain in the limit

$$\frac{dp}{u^np^r} + \frac{\mathrm{W}}{\mathrm{A}g}\,\frac{du}{u^np^r} + f\frac{\mathrm{S}}{\mathrm{A}}dx = 0. \tag{9}$$

But
$$u^np^r = u_0^np_0^np^{r-n} = u_0^rp_0^ru^{n-r};$$

thus
$$\frac{p^{n-r}dp}{p_0^nu_0^n} + \frac{\mathrm{W}}{\mathrm{A}g}\,\frac{u^{r-n}du}{p_0^ru_0^r} + f\frac{dx}{m} = 0,$$

where $m = \mathrm{A}/\mathrm{S}$ is the "hydraulic mean depth" of the pipe. Integrating, we get

$$\frac{p_0^{n-r+1}-p_1^{n-r+1}}{(n-r+1)p_0^nu_0^n} + \frac{\mathrm{W}}{\mathrm{A}g}\,\frac{u_0^{r-n+1}-u_1^{r-n+1}}{(r-n+1)p_0^ru_0^r} - f\frac{l}{m} = 0. \tag{10}$$

It is usual to omit the middle term in this equation as being very small compared with the other two terms; and as this is undoubtedly so in the present experiments, we may write

$$fl/m = (p_0^{n-r+1}-p_1^{n-r+1})/(n-r+1)\,p_0^nu_0^n. \tag{11}$$

If, now, we write w for the mean volume of gas passing through the tube per second at the mean pressure $\frac{1}{2}(p_0+p_1)$ or p_m, we have $w = \mathrm{A}u_m$, where u_m is the mean velocity in the tube or, as we shall here use it, the mean velocity of the gas at the section of the tube where the pressure is p_m; hence $p_0 u_0 = p_m u_m = p_m w/A$, and finally

$$f = \frac{mA^n}{l(n-r+1)} \cdot \frac{p_0^{n-r+1} - p_1^{n-r+1}}{p_m^n w^n}, \tag{12}$$

an equation resulting from the above hypotheses and deduced in the usual manner, connecting the coefficient of friction f with the flux through the pipe, to be compared with the experimental results. It is identical with the usual equation (4) if we write $n = 2$ and $r = 1$, and the reason for its deduction is to see whether these values of n and r are the most consistent with these experiments. How far the numbers obtained by applying this equation to the experimental results are influenced by the dimensions, etc., of the particular pipe used is not clearly known, and the method of deducing the equation itself is not free from objection, but as it is the usual method it has been adopted.

4. *Determination of the Essential Dimensions of the Appliances.*

Length of Experimental Tube.—The circumference of the coil on the drum was measured by wrapping thick drawing paper round the cylinder and correcting the outside diameter thus obtained for the tube radius to obtain the mean diameter of the coil. The number of coils was counted and the total lengths of coil between the pressure points were 108·8 feet in the coil used by Dr. Grindley and 108·2 feet in that used by Mr. Gibson.

Determination of the Internal Diameter of the Tube.—The method adopted was to take two short lengths (about 1 foot) from the beginning of the tube before winding and two similar lengths from the other end after winding the tube on the drum. From the mean external diameter and the weight of each length the mean internal diameter was deduced and found to be 0·1276 inch in the tube used by Dr. Grindley and 0·1248 inch for the second tube used by Mr. Gibson.

To check this method, after the experiments were concluded the second tube was flooded eight times with water, the volume of water occupying the tube being carefully measured in each case, and the diameter by this method was found to be 0·1246 inch at 62° F.

From these figures we obtain for a temperature of 62° F., for the first tube $\pi a^4/8l = \text{antilog}\ \overline{12}{\cdot}4597$ and for the second tube $\pi a^4/8l = \text{antilog}\ \overline{12}{\cdot}42105$.

At any other temperature the value of this factor, $\pi a^4/8l$, requires a correction for the temperature difference from 62° F.

To simplify calculation, equation (1) was put in the form

$$u = C\,(p_0 - p_1)/w,$$

and the constant C determined so that the values of $p_0 - p_1$ in inches of mercury at 32° and w in cubic feet per second, as given directly by the experiments, could be inserted at once in the equation to give μ in foot-pound-weight-second units. Thus with the first tube $C = \text{antilog } \overline{10}{\cdot}30876$ at 32° F. and with the second tube $C = \text{antilog } \overline{10}{\cdot}27006$ at 32°, with an addition of 0·0000202 to the logarithm for each degree difference of temperature from 32°.

5. *Method of Conducting the Experiments.*

The first series of experiments, viz., those numbered 1 to 68, were made by Dr. Grindley and were carried out with the double purpose of finding the coefficient of viscosity of the air, and of determining the critical velocity for the tube by finding approximately the pressure difference at the two ends of the tube necessary to produce unsteady motion of the air. In the majority of these experiments, the results of which are plotted in fig. 3 and given in Table I, the velocities of flow were above the critical, and for the determination of the coefficient of viscosity they were of little use. They showed, however, the difference of pressure below which it was necessary to work in order that the velocity of flow should be below the critical, and, further, gave information on the resistance to flow at velocities above the critical. The experiments were therefore continued by Mr. Gibson, the majority of these being carried out at velocities well below the critical. Of this second series of experiments, the results of those above the critical velocity are shown in fig. 3 and in Table II, while the results of those below the critical point are given in Tables III, IV, V, and VI.

In every experiment, the procedure was as follows:—The coil and reservoirs having been levelled, the water bath was raised to the required temperature. This temperature being steady, after the expiration of about 15 minutes the regulating cocks on both air reservoirs were adjusted so as to give as nearly as possible the required pressure difference along the pipe. The temperature in the high-pressure air vessel was then noted until it became steady.

Having got all the temperatures and pressures, as well as the pressure difference between the two ends of the pipe (measured by the differential gauge) constant, three readings of the water level in each of the two air vessels were taken at intervals of 30 seconds. The temperatures of the air in vessel No. 1, and of the external air, as well as of the water bath, were then taken, and from this time to the end of the experiment the pressures in the two reservoirs and the pressure difference at the two ends of the tube were read at intervals of three minutes.

The temperatures of the external air, of the air in the high-pressure reservoir, and of the coil, were taken at intervals of nine minutes, being practically constant throughout the experiment. The relative and absolute pressures in the two reservoirs were regulated as necessary during the experiment by adjustment of the water inlet and outlet valves. In every case the experiment was terminated when the temperature in the high-pressure reservoir was identical with that at the start. Three readings of the water level in each of the air vessels were then taken at intervals

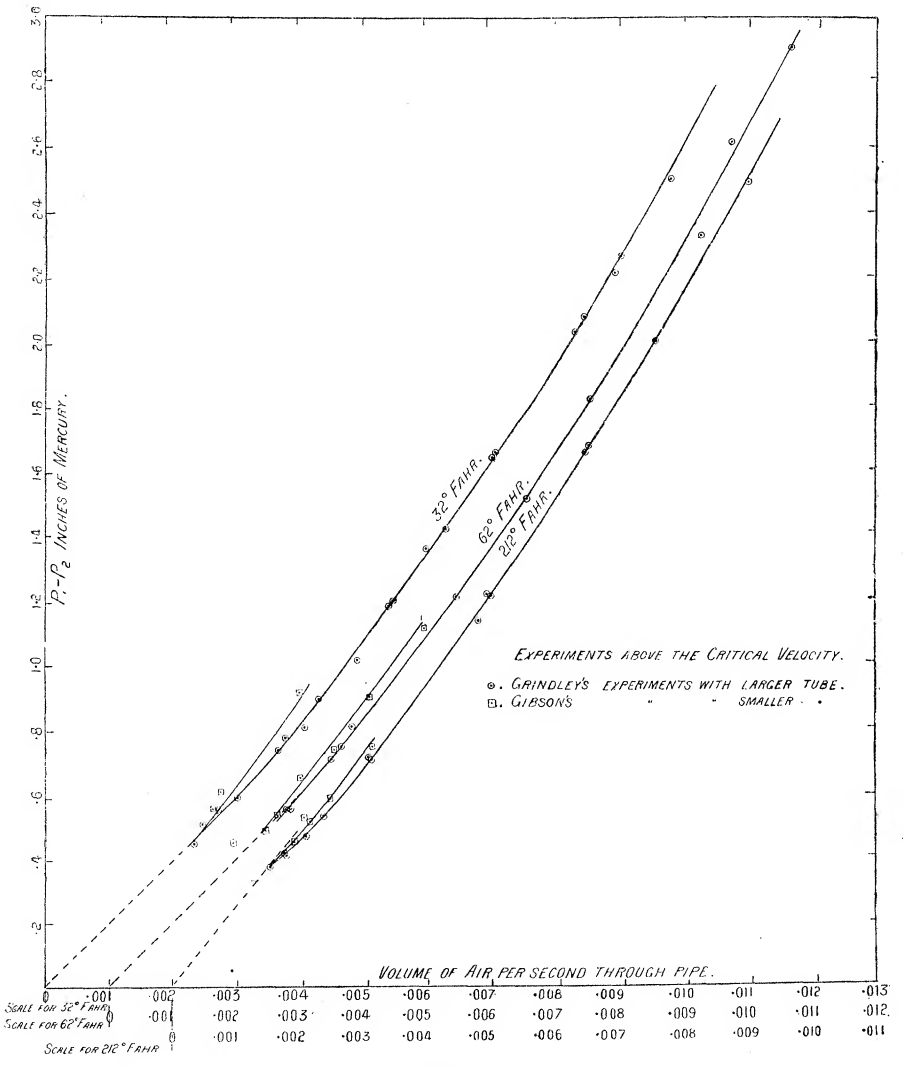


Fig. 3.

Experiment 172.—December 18, 1905.

Time.	Pressures.								Temperatures.			Volumes of water.	
	Reservoir I (ins. Hg).		Reservoir II (ins. Hg).		p_1 (ins. Hg).		(p_0-p_1) in inches of water multiplied by the gauge factor 1/0·868.		Air. ° F.	Reservoir I. ° F.	Coil. ° F.	Reservoir I.	Reservoir II.
2.50	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·70 + 1·44	4·14	66·8	68·0	163·2	1·410 1·413 1·417	2·310 2·314 2·318
2.53	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·71 + 1·45	4·16					
2.56	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·71 + 1·45	4·16					
2.59	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·71 + 1·45	4·16	66·3	68·0	162·6		
3.02	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·71 + 1·45	4·16	—	—	—	Barometer	30·169
3.05	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·71 + 1·45	4·16					
3.08	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·71 + 1·45	4·16	66·1	68·0	162·5		
3.11	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·71 + 1·45	4·16					
3.14	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·71 + 1·45	4·16					
3.17	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·70 + 1·44	4·14	65·9	68·0	161·7		
3.20	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·69 + 1·43	4·12	—	—	—	Barometer	30·167
3.23	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·70 + 1·44	4·14					
3.26	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·69 + 1·44	4·13	66·3	68·0	162·3		
3.29	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·69 + 1·43	4·12	—	—	—	Barometer	30·167
3.32	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·69 + 1·44	4·13					
3.35	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·69 + 1·44	4·13	67·2	68·0	162·5		
3.38	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·69 + 1·44	4·13					
3.41	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·69 + 1·44	4·13					
3.44	0·55 − 0·05	0·50	0·00 − 0·00	0·00	0·06 − 0·04	0·02	2·70 + 1·44	4·14	66·7	68·0	162·2	1·732 1·735 1·738	1·984 1·980 1·976
Totals	—	—	—	—	—	—	—	—	—	—	—	0·3217	
Means	—	0·50	—	0·00	—	0·02	—	4·143	66·4	68·0	162·4	0·3217/54	
Corrected for scale errors		0·50	—	0·00	—	0·02	—	—	66·6	68·8	161·2		
Corrected for temperatures and absolute pressures		30·58	—	30·10	—	30·12	—	—	—	—	—	Barometer	30·100

$$\frac{p_0+p_1}{2} = 30·26 \text{ ins. Hg.}$$

$$p_0-p_1 = \frac{4·143 \times 0·868}{13·596 \times 1·0013} = 0·2654 \text{ ins. Hg.}$$

$$\therefore\ \mu = 0·000000425 \text{ foot-pound-weight-second unit.}$$

$$w = \frac{0·3217 \times 30·58 \times 622·2}{54 \times 60 \times 30·26 \times 529·8}$$

$$= 0·0001170 \text{ cubic foot second.}$$

of 30 seconds. The length of time occupied in each experiment varied with the velocity of flow, from 30 minutes to two hours, the average time being about one hour.

A few experiments carried out above the critical velocity in the second tube indicated the similarity of the law governing unsteady flow in the two tubes, while the results of these being plotted with those of a second series of experiments carried out with low-pressure differences (see fig. 3) indicated that a pressure difference of about 0·4 inch of mercury (5·4 inches of water), was necessary to produce unsteady motion. After this, all the experiments were carried out with pressure differences of about 4 inches of water. In the latter series of experiments the volume of air passing per second was measured from the high-pressure reservoir alone, this being found to give perfectly satisfactory results.

A typical experiment is here given. From the experimental figures, the mean values of $p_0 - p_1$; p_0; p_1; and $\frac{1}{2}(p_0 + p_1)$ or p_m were obtained and corrected for scale errors, temperature, and the barometric height.

The mean volume passing through the tube at the mean pressure p_m and coil temperature was calculated from the mean of the volumes obtained from Reservoirs I and II at their respective mean pressures and temperatures, and in the latter series of experiments, from the readings from No. I reservoir alone. The flow per second is w, = V/(duration of experiment) in cubic feet.

6. *Results of the Experiments.*

The primary results required from each experiment are values of the pressures p_0 and p_1 at the ends of the experimental tube, w the quantity flowing through the tube at the mean pressure $\frac{1}{2}(p_0 + p_1)$ or p_m, and the temperature of the air in the tube. In the following Tables I to VI are placed values of $p_0 - p_1$ and p_m, from which p_0 and p_1 can be calculated for each experiment, together with the flux w, the temperature and the calculated value of μ obtained from equation (1). Values of μ so obtained have been given from experiments when the velocity of flow is above or near the critical velocity; such values, however, have only been used to assist in determining the critical velocity, and they have been placed in italics.

Table I.—Results of Dr. Grindley's Experiments, made for the most part at Velocities of Flow above the Critical Velocity.

No. of experiment.	Temperature.	$p_0 - p_1$ (ins. Hg).	$\frac{1}{2}(p_0 + p_1)$.	w (cub.ft.per sec.).	μ (foot-pound-weight-second units).
6	63·5	0·556	29·92	0·000277	*0·000000409*
5	63·5	0·557	36·05	282	*403*
8	61·5	0·710	30·04	347	*418*
9	61·5	0·747	37·55	362	*422*
4	63·5	0·804	29·98	377	*435*
12	66·2	1·208	30·58	541	*456*
10	60·2	1·488	31·04	649	
13	66·2	1·824	31·03	749	
11	60·2	2·324	31·50	924	
2	66·5	2·616	35·43	970	
14	69·2	2·910	31·96	0·001073	
3	66·5	3·411	33·36	1200	
15	69·2	4·185	32·92	1422	
16	74·0	4·892	33·06	1584	*0·000000630*
7	63·5	5·421	32·61	1725	*642*
1	66·5	5·474	31·08	1767	*633*
26	73·7	0·367	30·52	0·000208	*0·000000359*
23	73·5	0·464	30·58	244	*387*
25	73·5	0·785	30·89	385	*415*
18	73·5	0·936	30·84	443	*430*
24	73·5	1·140	31·28	519	
27	75·6	1·149	31·36	512	
17	73·3	1·304	31·23	584	
20	74·1	1·807	31·87	744	
29	75·7	1·812	32·15	730	
31	76·5	1·964	32·52	798	
28	75·5	2·143	32·71	845	
21	74·5	2·584	32·98	986	
33	77·0	2·685	33·70	0·001012	
30	75·9	2·820	33·71	1051	*0·000000547*
32	76·8	3·051	34·20	1117	*557*
34	77·0	3·217	34·57	1171	*560*
51	32·0	0·444	30·58	0·000237	*0·000000380*
52	32·0	0·586	30·63	302	*395*
46	32·0	0·734	30·82	364	*410*
43	32·0	0·774	30·70	373	*422*
35	32·0	0·800	30·64	405	
47	32·0	0·887	30·86	427	
38	33·0	1·059	30·79	489	
44	32·0	1·173	31·08	532	
50	32·0	1·190	31·00	541	
41	32·0	1·358	31·07	592	
54	32·0	1·418	31·08	627	
45	32·0	1·637	31·31	699	
36	32·0	1·653	31·13	703	
40	32·0	2·036	31·47	829	
48	32·0	2·070	31·57	839	
49	32·0	2·218	31·64	890	
53	32·0	2·262	31·57	895	*0·000000514*
37	32·0	2·498	31·64	974	*522*
42	32·0	2·733	31·87	0·001030	*540*

Table I—*continued.*

No. of experiment.	Temperature.	$p_0 - p_1$ (ins. Hg).	$\frac{1}{2}(p_0 + p_1)$.	w (cub. ft. per sec.).	μ (foot-pound-weight-second units).
68	212 ·0	0 ·3672	29 ·69	0 ·000161	*0 ·000000468*
64	212 ·0	0 ·4083	30 ·65	176	*476*
67	212 ·0	0 ·478	29 ·83	208	*472*
62	212 ·0	0 ·538	31 ·10	232	*475*
58	212 ·0	0 ·709	30 ·94	304	*479*
55	212 ·0	0 ·708	30 ·99	306	
65	212 ·0	1 ·135	30 ·18	473	
56	212 ·0	1 ·217	31 ·25	489	
61	212 ·0	1 ·215	31 ·14	494	
57	212 ·0	1 ·652	31 ·43	641	
59	212 ·0	1 ·672	31 ·38	647	*0 ·000000530*
63	212 ·0	1 ·993	31 ·57	749	*546*
60	212 ·0	2 ·484	31 ·71	898	*567*

Table II.—Results of Mr. Gibson's Experiments above the Critical Velocity.

No. of experiment.	Temperature.	$p_0 - p_1$ (ins. Hg).	$\frac{1}{2}(p_0 + p_1)$.	w.
118	32 ·2° F.	0 ·513	30 ·00	0 ·000255
119	32 ·2	0 ·553	29 ·69	270
120	32 ·2	0 ·5585	35 ·77	263
121	32 ·0	0 ·9105	30 ·91	398
122	32 ·0	0 ·603	30 ·78	274
71	60 ·0	0 ·454	29 ·99	0 ·000193
72	60 ·0	0 ·4815	30 ·34	243
73	60 ·0	0 ·528	29 ·51	303
74	60 ·0	0 ·538	30 ·03	260
75	60 ·0	0 ·6415	29 ·08	298
76	60 ·0	0 ·724	30 ·55	354
77	60 ·0	0 ·892	30 ·37	405
78	60 ·0	1 ·104	30 ·06	489
79	60 ·0	1 ·544	31 ·06	630
104	212 ·0	0 ·455	26 ·54	0 ·000188
105	212 ·0	0 ·455	29 ·99	191
106	212 ·0	0 ·419	30 ·48	212
107	212 ·0	0 ·588	40 ·01	242
108	212 ·0	0 ·741	27 ·15	308

Table III.—Results of Mr. Gibson's Experiments below the Critical Velocity.

No. of experiment.	Temperature.	p_0-p_1 (ins. Hg).	$\frac{1}{2}(p_0+p_1)$.	w (cubic feet per second).	μ (in foot-pound-weight-second units).
123	32·2° F.	0·1740	30·00	0·0000907	0·000000357
124	32·1	0·2400	30·05	1253	356
125	32·1	0·1883	30·02	0982	357
126	32·1	0·2984	30·08	1582	351
127	32·1	0·2997	30·09	1550	360
128	32·0	0·2240	29·90	1230	352
129	32·0	0·1879	29·90	1017	357
130	32·0	0·1282	29·84	0710	351
131	32·0	0·4378	29·69	2290	356
132	32·0	0·2413	29·83	1266	355
133	32·0	0·0904	29·66	0470	358
134	32·0	0·1994	30·09	1047	355
135	32·0	0·1399	30·30	0756	356
136	32·0	0·2041	26·80	1100	358
137	32·0	0·1956	26·12	1067	354
138	32·0	0·2025	30·13	1113	351

Mean value of $\mu = 0·0000003553$.

Greatest positive variation from mean .... 1·35 per cent.
" negative " " .... 1·15 "

Table IV.

No. of experiment.	Temperature.	p_0-p_1 (ins. Hg).	$\frac{1}{2}(p_0+p_1)$.	w (cubic feet per second).	μ (foot-pound-weight-second units).
80	60·3° F.	0·2707	29·90	0·0001346	0·000000375
81	60·0	0·3147	29·83	1591	369
82	60·0	0·3069	29·84	1535	373
83	60·2	0·3273	29·75	1633	374
84	60·0	0·3377	29·85	1685	374
85	59·0	0·2414	28·49	1263	369
86	58·8	0·2786	34·83	1465	369
87	58·0	0·2140	34·61	1102	371
88	60·0	0·0606	29·66	0301	375
89	60·0	0·1149	29·85	0600	370
90	60·0	0·1586	30·04	0815	374
91	60·0	0·2357	29·16	1225	372
92	60·0	0·2418	30·06	1260	371

Mean value of $\mu = 0·0000003720$.

Greatest positive variation from mean .... 0·91 per cent.
" negative " " .... 0·81 "

Table V.

No. of experiment.	Temperature.	$p_0 - p_1$ (ins. Hg).	$\frac{1}{2}(p_0 + p_1)$.	w (cubic feet per second).	μ (foot-pound-weight-second units).
93	212° F.	0 ·2153	30 ·04	0 ·0000907	0 ·000000446
94	212	0 ·2301	40 ·35	0966	447
95	212	0 ·2512	29 ·12	1049	450
96	212	0 ·2578	39 ·82	1082	447
97	212	0 ·2597	30 ·81	1082	451
98	212	0 ·2945	30 ·48	1237	447
99	212	0 ·2978	29 ·96	1115	447
100	212	0 ·3118	42 ·37	1320	443
101	212	0 ·3387	44 ·26	1425	446
102	212	0 ·4616	40 ·29	1940	447
103	212	0 ·4552	30 ·95	1908	448

Mean value of μ = 0 ·0000004473.

Greatest positive variation from mean .... 0 ·79 per cent.
" negative " " .... 0 ·85 "

Table VI.

No. of experiment.	Temperature.	$p_0 - p_1$ (ins. Hg).	$\frac{1}{2}(p_0 + p_1)$.	w (cubic feet per second).	μ (foot-pound-weight-second units).
109	35 ·0° F.	0 ·2024	30 ·33	0 ·0001058	0 ·000000356
110	36 ·5	0 ·2025	29 ·96	1080	349
111	37 ·5	0 ·1015	30 ·02	0529	357
112	38 ·0	0 ·1705	37 ·77	0883	360
113	39 ·4	0 ·3424	29 ·78	1770	360
114	40 ·7	0 ·1927	29 ·68	0956	361
115	41 ·2	0 ·3072	29 ·92	1560	367
116	41 ·4	0 ·3065	29 ·92	1573	363
117	42 ·9	0 ·3086	29 ·90	1580	364
139	48 ·5	0 ·3005	29 ·88	1543	363
140	53 ·8	0 ·3034	29 ·69	1530	369
141	54 ·0	0 ·3288	29 ·67	1663	369
142	54 ·0	0 ·2942	29 ·73	1485	369
144	74 ·1	0 ·3406	30 ·08	1670	381
145	74 ·7	0 ·1773	29 ·82	0866	382
146	77 ·5	0 ·3068	30 ·03	1492	384
147	77 ·5	0 ·3451	30 ·07	1686	382
149	91 ·2	0 ·3297	30 ·13	1590	387
150	91 ·4	0 ·3023	30 ·01	1471	384
151	95 ·6	0 ·3504	29 ·93	1656	395
152	96 ·2	0 ·2513	29 ·87	1196	393
153	95 ·7	0 ·3425	29 ·84	1646	389
154	98 ·7	0 ·3031	29 ·81	1446	391
155	118 ·7	0 ·2977	29 ·77	1377	404
156	120 ·0	0 ·3057	29 ·74	1448	395
157	119 ·8	0 ·2878	29 ·72	1337	403
158	121 ·9	0 ·3189	29 ·21	1474	405
159	120 ·1	0 ·3334	30 ·33	1542	404
160	120 ·0	0 ·2199	30 ·28	1002	411
161	134 ·1	0 ·3178	30 ·31	1446	411
162	134 ·4	0 ·3160	30 ·14	1439	411
163	135 ·2	0 ·3107	29 ·81	1407	413

Table VI—*continued.*

No. of experiment.	Temperature.	$p_0 - p_1$ (ins. Hg).	$\frac{1}{2}(p_0 + p_1)$.	w (cubic feet per second).	μ (foot-pound-weight-second units).
164	135·9° F.	0·3241	29·82	0·0001460	0·000000415
165	141·8	0·2894	30·27	1308	414
166	142·5	0·3197	30·19	1429	419
167	143·9	0·3229	30·30	1454	416
168	150·3	0·3179	29·70	1417	420
169	151·4	0·3390	30·31	1486	423
170	151·7	0·3084	29·68	1383	418
171	150·9	0·2873	30·66	1302	413
172	161·2	0·2655	30·26	1170	425
173	163·0	0·3312	30·30	1457	426
174	163·5	0·2982	30·28	1294	431
175	167·7	0·2965	31·53	1281	433
176	175·0	0·2392	30·20	1244	436
177	178·8	0·3786	29·68	1636	434
178	179·1	0·3070	29·64	1346	428
179	180·0	0·3116	30·29	1336	437
180	182·1	0·3080	30·13	1322	437
181	185·4	0·2849	30·04	1221	438
182	198·6	0·2973	30·28	1258	444
183	199·2	0·2943	30·28	1243	444
184	204·0	0·3290	30·19	1383	447
185	204·7	0·2929	29·99	1234	445

7. *The Determination of the Critical Velocity.*

The critical velocity of flow through the pipe at any particular temperature can best be obtained by calculating the value of μ from the figures given in Table I, on the assumption that equation (1) holds for these experiments, and comparing these incorrect values of the coefficient of viscosity with the correct values given by Tables III, IV, and V. If all these values of μ are plotted with the flux w, it should be easy to determine the value of w at which the value of μ begins to increase.

It would appear that the value of w at the critical velocity was between 0·00017 and 0·00018 cubic feet per second, which gives for the critical velocity a mean value of approximately 1·95 feet per second. Its accurate determination was difficult, but the results indicate that for this pipe the critical velocity lies between 1·9 and 2·0 feet per second.

The results also show that if there is any variation in the value of this critical velocity with temperature it cannot be very great.

8. *The Coefficient of Viscosity and its Variation with Temperature and Pressure.*

A consideration of the kinetic theory of gases led Maxwell* to the conclusion that the coefficient of viscosity of a gas would be independent of the pressure. This deduction, verified by many observers,* receives additional confirmation from the results of Experiments 94, 96, 100, 101, 102, and of Experiments 93, 95, 98, 99.

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* 'Phil. Mag.,' 1860, vol. 19, p. 20.

These are as follows :—

No. of experiment.	Temperature.	$\frac{1}{2}(p_0+p_1)$ inches of mercury.	μ (foot-pound-weight-second units).
	° F.		
94	212	40·3	0·000000447
96	212	39·8	447
100	212	42·4	444
101	212	44·2	446
102	212	40·3	447
Mean value of μ = 0·0000004462			
93	212	30·0	0·000000446
95	212	29·1	450
98	212	30·5	447
99	212	30·0	447
Mean value of μ = 0·0000004475			

A comparison of these results tends to show that, within the limits of experimental error, the viscosity is independent of the pressure.

To test the variation of μ with temperature in the present experiments, values of $\log\mu$ and of $\log\tau$ have been plotted (fig. 4). From the fact that the plotted points lie on a regular curve and not on a straight line, it is evident that no such law as μ and τ^n, where n is a constant, holds for the range of temperature from $\tau = 493°$ abs. to $\tau = 673°$ abs.

The variation of n in this formula, with temperature, is shown in the following table :—

Temperature range.		n.
Absolute.	F.	
493°—521°	32°— 60°	0·830
521 —551	60 — 90	0·811
551 —581	90 —120	0·788
581 —611	120 —150	0·759
611 —641	150 —180	0·665
641 —673	180 —212	0·561

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* See the 'Physical Review,' vol. 24, p. 335, for a *résumé* of much of the work done in this connection.

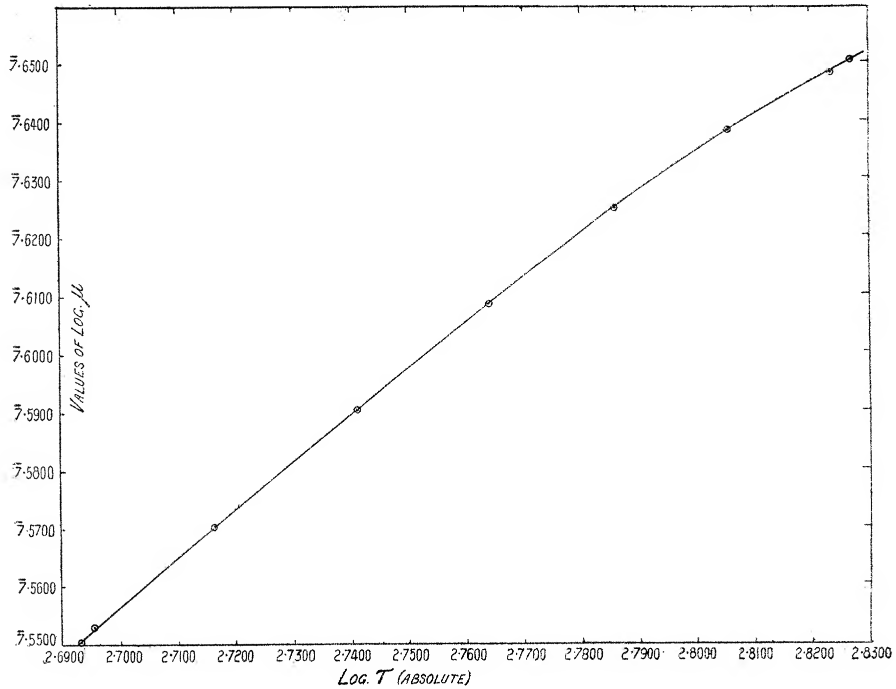

FIG. 4.

Rayleigh,* using the transpiration method, found n to have a mean value of 0·754 between the temperature limits 15° and 100° C. Between the same limits the present experiments show a mean value for $n = 0{\cdot}727$.

Sutherland† has proposed, and has justified on theoretical grounds, the formula

$$\mu = \frac{\mathrm{K}\tau^{\frac{1}{2}}}{1+\mathrm{C}\tau^{-1}},$$

where C and K are constants, and where τ is the absolute temperature, this relation holding for temperatures above the critical within the range of pressures over which Boyle's law is approximately true.

Writing this in the form $\tau = \mathrm{K}\tau^{\frac{3}{2}}/\mu - \mathrm{C}$, we have a linear equation in τ and $\tau^{\frac{3}{2}}/\mu$. Values of τ and of $\tau^{\frac{3}{2}}/\mu$, as calculated from the present series of experiments, have been plotted in fig. 5, and, by their close coincidence with a straight line, confirm this formula.

The values of K and C most nearly fitting the above equation have been deduced from the results of the whole of the present series of experiments by the method of least squares, the straight line shown in fig. 5 involving these values.

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* 'Roy. Soc. Proc.,' 1900, vol. 66, p. 68.

† 'Phil. Mag.,' 1893, vol. 36, p. 507.

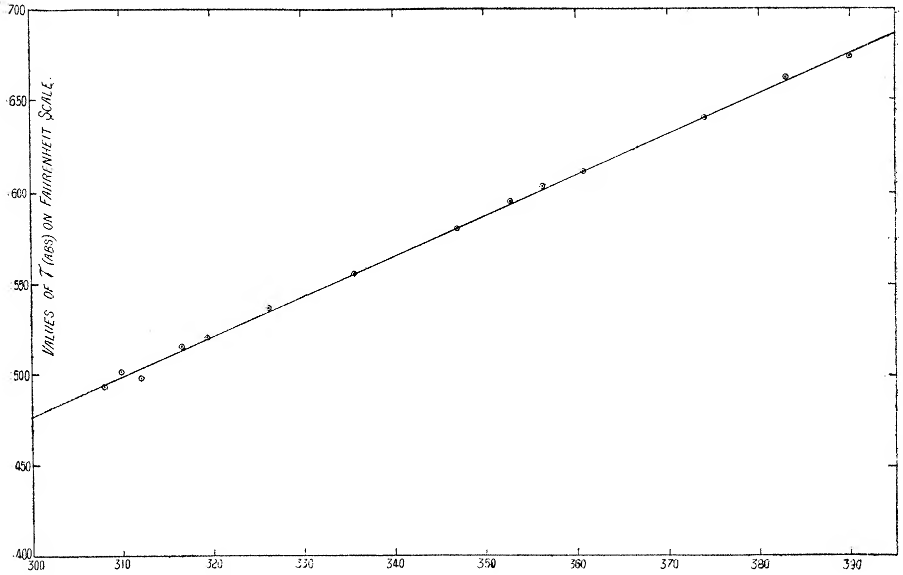

FIG. 5.

These are as follows, the abscissa representing $\tau^{\frac{3}{2}}/10^7\mu$.

	K.	C.
τ measured on F. scale (0° F. = 461 abs.); μ in foot-pound-weight-second units	$220·7 \times 10^{-10}$	184·5
τ measured on C. scale (0° C. = 274 abs.); μ in foot-pound-weight-second units	$296·0 \times 10^{-10}$	102·5
τ measured on C. scale (0° C. = 274 abs.); μ in absolute foot-pound-weight-second units	$952·0 \times 10^{-9}$	102·5
τ measured on C. scale (0° C. = 274 abs.); μ in absolute C.G.S. units	$141·8 \times 10^{-7}$	102·5

The plotted points in fig. 6 show the experimental variation of viscosity with temperature, while the full line curve shows the graph of

$$\mu = \frac{220·7 \times 10^{-10}\tau^{\frac{1}{2}}}{1+184·5\tau^{-1}}.$$

The maximum experimental deviation from this law is 1·3 per cent.

The only other experiments known to the authors, from which the value of C has been determined, are those of Holman* and of Barus.† In those of Holman, over a temperature range of 14° to 124°·4 C., C was found to have the value 113, the units being on the absolute C.G.S. scale, while in the experiments of Barus the value of C, with the same units, and between the temperature limits 442° and 1216° C., was found to be 102, as compared with 102·5 in the present experiments.

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* 'Phil. Mag.,' 5th series, 1886, vol. 21, p. 199.

† 'Am. Journ. Sci.,' 3rd series, vol. 135.

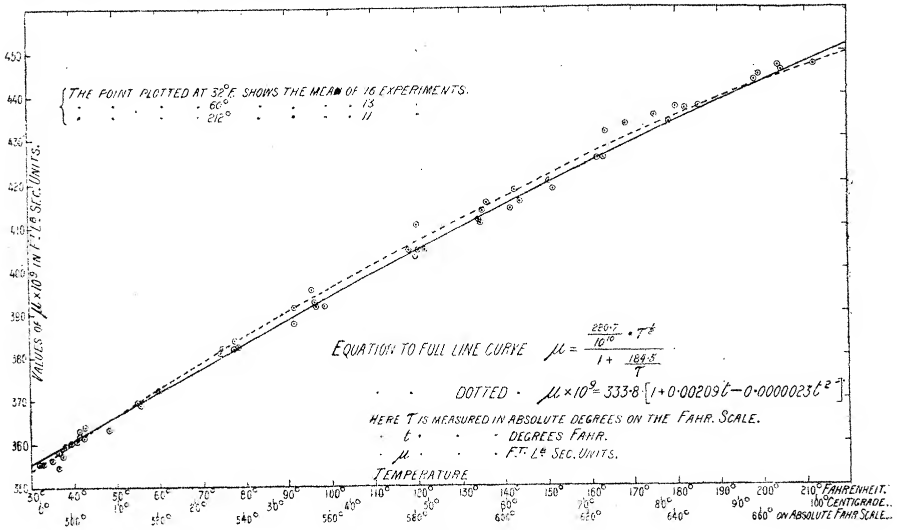

FIG. 6.

For comparison with the published results of other experimenters, the results of the present experiments have been expressed in the form

$$\mu = a\{1+bt+ct^2\}.$$

The constants in this equation, the graph of which is shown in dotted lines in fig. 6, are as follows, t being measured in degrees Centigrade :—

	a.	b.	c.
μ in foot-pound-weight-second units	$333{\cdot}8\times10^{-9}$	0·00209	0·0000023
" " " "	$355{\cdot}3\times10^{-9}$	0·00329	0·0000070
μ in absolute foot-pound-second units	$114{\cdot}2\times10^{-7}$	0·00329	0·0000070
" C.G.S. units	$170{\cdot}2\times10^{-6}$	0·00329	0·0000070

as compared with Holman's,

$$\mu = 171{\cdot}55\,\{1+0{\cdot}002751t-0{\cdot}00000034t^2\}10^{-6} \text{ abs. C.G.S. units.}$$

Tomlinson,[*] with a temperature range of from 9°·97 C. to 14°·63 C., adopted this same law of temperature variation. The value of μ at Tomlinson's mean temperature, 11°·8 C., as calculated by the author's formula, agrees within one-tenth of 1 per cent. with the value as experimentally determined by Tomlinson.

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[*] 'Phil. Trans.,' 1886, part 2, p. 767.

The following table, showing values of μ at 0° C., as obtained by various observers, may be of interest :—

Observer ……	Meyer.*	Puluj.*	Schneebeli.†	Obermayer.†	Holman.	Tomlinson.	The authors.
Value of μ at 0° C.	0·0001727	0·0001798	0·0001707	0·0001705	0·0001715	0·0001715	0·0001702

9. *The Determination of the Frictional Resistances at Velocities of Flow greater than the Critical.*

As the experiments proceeded it was observed that the quantity of air through the tube in any experiment could not be estimated previously by calculation from known formulæ containing empirical constants; and hence the experimental figures were used to obtain, by using the usual equation (4), p. 119, values of the coefficient ζ. Now, if the laws of resistance are in reality those used in demonstrating equation (4), it follows that ζ should be practically a constant for experiments with air in the same tube and at the same constant temperature, unless the influence of tube diameter is far greater than at present suspected.

A few values of ζ, calculated by using equation (4), are here given :—

No. of experiment.	ζ.	u_m—the mean velocity of air in the tube in feet per second.	p_m—the mean pressure in the tube in inches Hg.
6	0·0837	3·12	29·92
5	0·0675	3·18	36·05
9	0·0529	4·07	37·55
4	0·0654	4·25	29·98
12	0·0467	6·09	30·58
13	0·0365	8·43	31·03
2	0·0272	10·9	35·43
3	0·0246	13·5	33·37
7	0·0194	19·4	32·61
1	0·0196	19·9	31·08

These values of ζ are considerably higher than those given by previous authors, which is possibly accounted for by the small diameter of the pipe, and a very large variation in its value with the mean velocity u_m is shown, the connection between ζ and u_m being shown by the curve in fig. 7.

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* 'Phil. Mag.,' 1886, vol. 21, p. 220.

† 'Archives Sci. Phys. Nat.,' 1885, vol. 14.

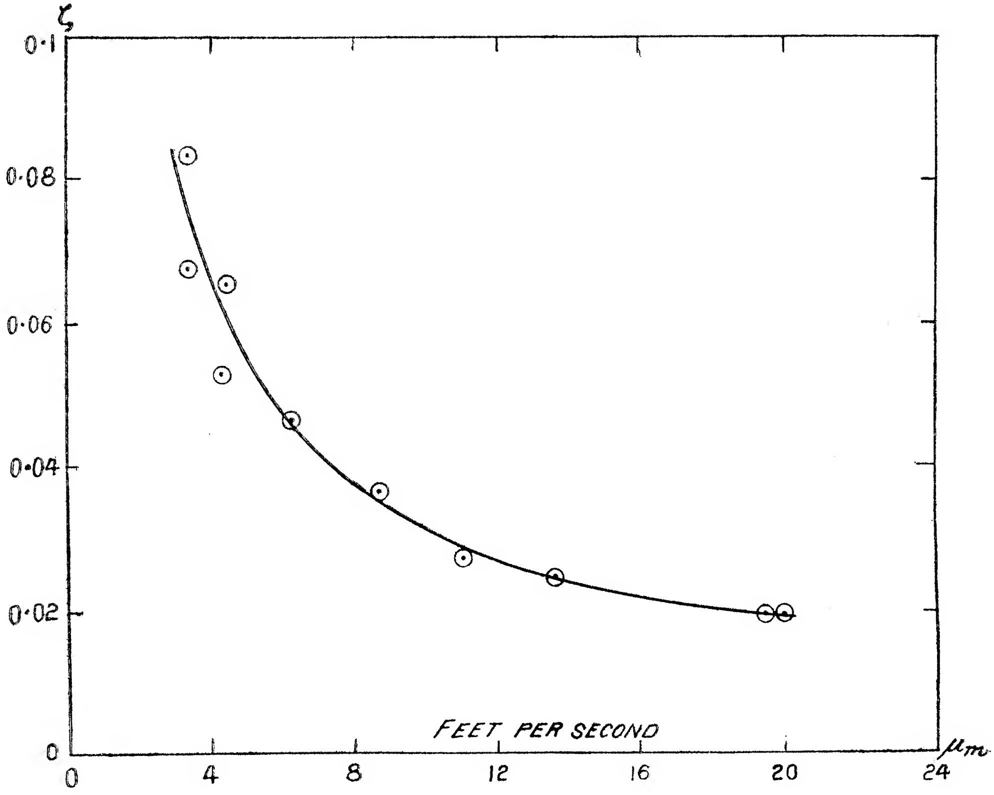

Fig. 7.

Unwin* deduces values of ζ from Riedler's experiments on a pipe $11\frac{3}{4}$ inches diameter, and gives varying values of ζ from 0·00184 to 0·00449. From Stockalper's experiments Unwin deduces the values $\zeta = 0·00449$ from a pipe 0·492 foot diameter, and $\zeta = 0·00377$ from a pipe 0·656 foot diameter, and he gives from these results a formula for calculating ζ for pipes of different diameters in form—

$$\zeta = 0·0027\left(1+\frac{3}{10d}\right).$$

For a pipe 1/8 of an inch diameter this formula would give $\zeta = 0·0805$, a value which would be obtained from the present experiments if the mean velocity u_m was a little over 3 feet per second. The variation in the value of ζ with u_m is, however, too great to permit of Unwin's formula being used for so small a pipe.

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* 'Min. Proc. I.C.E.,' vol. 105, p. 190.

It would be interesting to use the experimental figures to find whether the variation in the frictional resistances is best represented by u^2 or by some other power of u. An examination of the curve in fig. 7 will show that the points lying above or below the mean curve correspond to the value of p_m in the experiments they represent, being below or above an average value of p_m, and it apparently indicates that the frictional resistances may bear some unexpressed relation to the pressure in the gas.

Whether a pressure variation is real or not, it appears reasonable to assume that ζ should have a more constant value for the particular pipe used than that shown in the last table if the resistances do vary as the assumed square of the velocity. While the whole question can only be definitely settled by making experiments similar to the present ones on pipes of different diameters, still it is interesting and useful to observe what power of the mean velocity in the formula for the frictional resistances would give a constant value of a coefficient of friction for this pipe.

It was found impossible to obtain this constant value without reconstructing the primary formula as shown on pp. 120 and 121, and use was, therefore, made of the general equation (12), p. 121, which can be put in the form

$$f = \mathrm{C}\,.\,\frac{p_0^{\,n-r+1} - p_1^{\,n-r+1}}{p_m^{\,n}w^n}, \tag{13}$$

where C is a constant in any series of experiments on the same tube with air at the same constant temperature. Values of n and r were sought which would make the factor $(p_0^{\,n-r+1} - p_1^{\,n-r+1})/p_m^{\,n}w^n$ a constant during any such series of experiments, in which case f would also be constant.

The determination of these exponents involved much time and labour, but their values appear to be

$$n = 1{\cdot}25 \quad \text{and} \quad r = 0. \tag{14}$$

This value of n is not likely to be in error by as much as 1 per cent., and $r = 0$ is the most probable value of r, though any very small value of r would serve. This value of r is very peculiar, since it leads to the result that the frictional resistances are independent of the pressure in the gas; this point will be referred to later.

With these values of n and r, the values of $(p_0^{\,n-r+1} - p_1^{\,n-r+1})/p_m w$, as calculated from the results of the experiments above the critical velocity at any particular temperature, are practically constant; thus at a mean temperature of 75°·2 F. the greatest variation of the value of this factor from the mean is less than 2·9 per cent.

Using formula (13) to determine values of f, using $n = 1{\cdot}25$ and $r = 0$, we obtain the following mean results :—

$f = 0{\cdot}000215$ at temperature 32° F.
$f = 0{\cdot}000219$ „ 75°·2 F.
$f = 0{\cdot}000242$ „ 212° F.

the units being the foot-pound weight and second.

These values are plotted in fig. 8, showing that the variation in the value of f with temperature Fahr. t can be represented by

$$f = f_0(1+\alpha t+\beta t^2), \tag{15}$$

where $f_0 = 0{\cdot}000213$, $\alpha = 0{\cdot}000295$, and $\beta = 0{\cdot}0000017$.

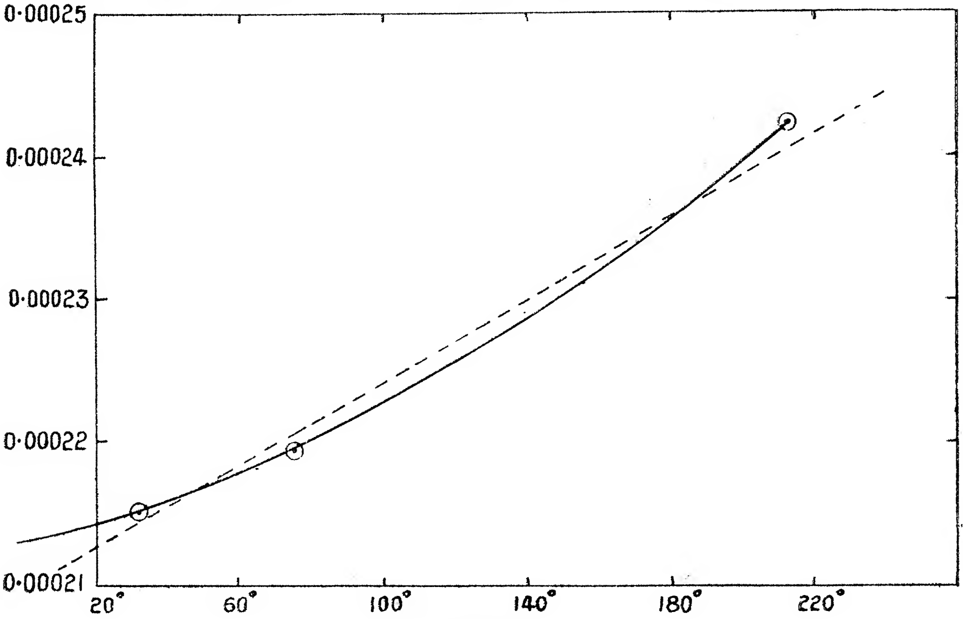

FIG. 8.

The chief conclusions, therefore, to be drawn from these numbers are that for this particular size of tube the frictional resistances are better represented by a velocity variation represented by $u^{1{\cdot}25}$ than by the usually accepted u^2, and, further, that the frictional resistances do not vary with the pressure of the air in the tube.

Though this variation of the frictional resistances with the velocity is unusual, and requires further evidence on pipes of other diameters before it could be accepted, it is worthy of note that if the formula (12), which can be put in the form

$$f = \frac{0{\cdot}0824}{l} \cdot \frac{d^{3\cdot5}}{w^{1\cdot25}} \cdot \frac{p_0^{2\cdot25}-p_1^{2\cdot25}}{p_m^{1\cdot25}}, \tag{16}$$

is applied to determine f, using the results of Riedler's experiments previously mentioned,* a mean value of f calculated from all the experiments there quoted is 0·000217, while the value of f calculated from (15) with $t = 50°$ would also be 0·000217.

From Stockalper's experiments, using the only figures given in the above paper, f would have a mean value 0·000182, but omitting one quite irregular experimental result we should get a mean value 0·000201.

These values are remarkably near those given from the present experiments, considering that the mean pressures in Riedler's experiments were about 100 lbs. per square inch, with velocities ranging from 10 to 24 feet per second (similar to those in the present experiments), and on a pipe $11\frac{3}{4}$ inches diameter.

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* 'Min. Proc. I.C.E.,' vol. 105.